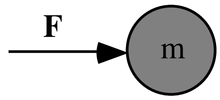
Crystal Momentum and Effective Mass

Consider a force **F** acting on a mass.



The energy ΔE added to the mass as it moves a distance Δs is:

$$\Delta E = F\Delta x ,$$

which yields

$$\frac{\partial E}{\partial x} = F$$

This can be re-arranged to read

$$\frac{\partial E}{\partial x} = \frac{\partial t}{\partial x}\frac{\partial E}{\partial t} = F$$

Noting that $\mathrm{v}_g = \frac{\partial x}{\partial t} = \frac{1}{\hbar}\frac{\partial E}{\partial k}$, we can write:

$$F = \frac{\partial t}{\partial x}\frac{\partial E}{\partial t} = \frac{1}{\mathrm{v}_g}\frac{\partial E}{\partial t} = \hbar\frac{\partial k}{\partial E}\frac{\partial E}{\partial t}$$

which yields:

$$\frac{\partial}{\partial t}(\hbar k) = F .$$

This means that k will increase linearly with time when a positive force F is applied to a mass. This is similar to the behavior of the momentum p of a free mass when a force F is applied: Since it increases with applied force like momentum, $\hbar k$ is a momentum-like variable, often called *crystal momentum*.

For the case of a free particle, $\frac{\partial p}{\partial t} = F$, where $p = \sqrt{2mE} = \hbar k$. which means that crystal momentum is actual momentum when a particle is free of lattice forces.

Effective Mass

Even though crystal momentum increases with applied force, it may well be that the particle velocity is decreasing. To see why, consider the time-rate-of-change of the group velocity of a particle:

$$a = \frac{\partial}{\partial t} \mathrm{v}_g = \frac{1}{\hbar} \frac{\partial}{\partial t} \frac{\partial E}{\partial k}$$

This can be rewritten as:

$$a = \frac{\partial}{\partial t} \mathrm{v}_g = \frac{1}{\hbar} \frac{\partial k}{\partial t} \frac{\partial^2 E}{\partial k^2}$$

But, since $\frac{\partial k}{\partial t} = \frac{F}{\hbar}$, we obtain:

$$F = \left[\frac{1}{\hbar^2} \frac{\partial^2 E}{\partial k^2} \right]^{-1} a = m^* a$$

where

$$m^* = \left[\frac{1}{\hbar^2} \frac{\partial^2 E}{\partial k^2} \right]^{-1}$$

is the *effective mass* of the particle.

The effective mass of a particle indicates how its velocity will change when a force is applied. If the particle's velocity increases in the direction of an applied force, then its effective mass is positive. If, on the other hand, its velocity decreases in the direction of an applied force, then its effective mass is negative. Contrary to common belief, a particle with a negative effective mass doesn't have a negative weight. Rather, it slows down when a force is applied, since its potential energy is increasing at the expense of its kinetic energy.